Waves and Oscilations

Caio Augusto Siqueira da Silva

March, 2022

1 Estimated time to solve - 1h15min

In this question, all oscillatory motions are assumed to be small.
As shown in Fig. 2, a string of mass m and length l with tension τ has a mass M attached to the end. The mass M can slide in a vertical direction on a frictionless rod at $x = l$. The shape of the string is described by a function $y(x, t)$. The string is fixed at the origin $y(0, t) = 0$

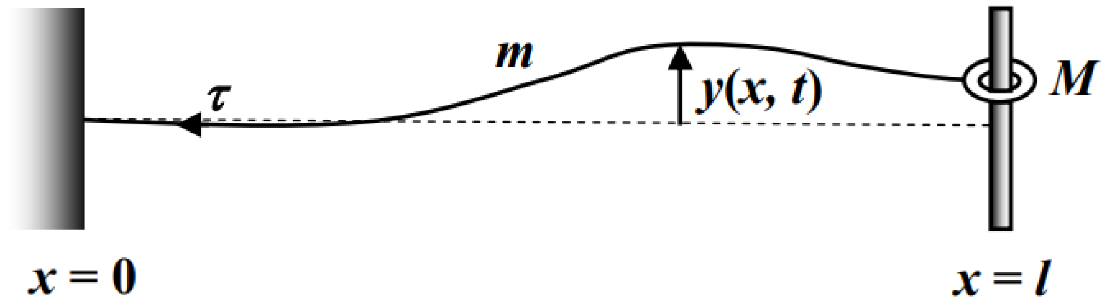


Figure 2 图二

1. First assume that mass M is held fixed at y = 0. Write down the general solution y(x, t) for the standing waves on the string. Express your answer in terms of the given parameters and arbitrary constants.
2. Now assume that mass M can slide up and down on a frictionless rod at x = l. What is the boundary condition on y(x, t) at x = l? You can assume that the oscillations are small.
3. Write down an equation for the frequencies of the standing waves on the string when the mass M is free to slide. You do not need to solve the equation.
4. If $m \ll M$, find the two lowest normal mode frequencies.
5. For $m \ll M$, calculate the ratio of the kinetic energy of the string to that of the mass M at the lowest normal mode frequency.
6. If $M \ll m$, find the two lowest normal mode frequency.
7. For $M \ll m$, calculate the ratio of the kinetic energy of the string to that of the mass M at the lowest normal mode frequency.
8. A traveling wave of angular frequency ω is generated near the end x = l. It propagates towards the mass M and is reflected with a phase shift of $\pi/2$. What is the value of ω in terms of τ, m, M and l?

2 Estimated time to solve - 1h30min

Quantum Optics Light has wave-particle duality, so light waves can be described by one dimensional oscillator in classical mechanics. Quantum optics utilizes the quantization of Electromagnetic field to describe the quantum state of photons by quantum state of one dimensional oscillator.

Part A Consider electromagnetic wave transmitting in a cubic cavity of side length L, and volume V, as the figure shows. The electric field is

$$E_x(z,t) = Esin(kz)sin(\omega t + \Phi)$$

where E_0 is the magnitude of the oscillation,$k = 2\pi/\lambda$ is the wave vector, ω is the angular frequency of the oscillation. Consider the case of $\Phi = 0$ in parts (i) to (iii)

1. the magnetic field in the cavity can be expressed as $B_y(z,t) = B_0 f(z,t)$. Find E_0/B_0 and the function $f(z,t)$

2. Find the total energy of the electromagnetic field in the cavity E(t)

3. Now introduce two functions $q(t)$ and $p(t)$ satisfies $q(t) \propto sin(\omega t)$, $p(t) \propto cos(\omega t)$. then E(t) can be rewritten as $E = \frac{1}{2}(p^2 + \omega^2 q^2)$. Find $p(t)$, $q(t)$, express them with V, E_0, ω
   P.S. Here, $p(t) = dq/dt$, $dp/dt = -\omega q(t)$. The energy equation above can be viewed as the energy of an oscillator.

   Part B In quantum mechanics, energy of an one-dimensional simple harmonic oscillator can be written as

   $$E_n = (n + \frac{1}{2})\frac{h}{2\pi}\omega$$

   Here, n can be interpreted as the number of photons, h is the Planck's constant. q and p satisfies Heisenberg's uncertainty principle, meaning that

   $$\Delta q \Delta p \leq \frac{h}{4\pi}$$

   is true. Now, rewrite the electric field as

   $$E_x(z,t) = \sqrt{\frac{2h\omega}{\epsilon_0 V}} \cdot E_0 sin(kz)\left(X_1(t)\cos\Phi + X_2(t)\sin\Phi\right)$$

   Where $X_1(t) = \eta q(t)$ and $X_2(t) = \zeta p(t)$

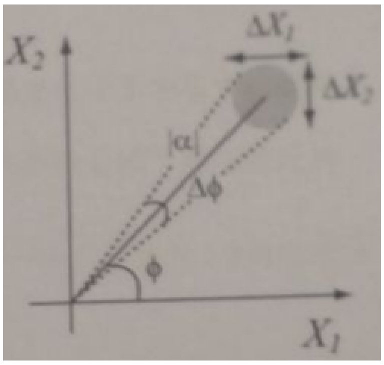
4. Find the parameters η. ζ

   In quantum optics we often use phasor diagram (i.e. X_1-X_2 graph) to describe the quadrature of the electric field. The quantum state of photons α can be expressed as $\alpha = X_1 + iX_2$ in the phasor diagram as shown on the next page.

5. Find $|\alpha|$, express it in terms of V, E_0, ω

6.) Consider the average energy of the electromagnetic wave, when $n \gg 1$, we can get $n = |\alpha|^a$. Find the constant a. Strictly speaking, we have to consider the average number of photons in the cavity, meaning n is expressed by $\bar{n}$.

7. From the quantum energy equation above, the system has energy $h\omega/4\pi$ even when there's no photon inside the cavity, this energy is called zero-point energy. We can say that the existence of zero-point energy is caused by vacuum fluctuation. If that's the case, find the electric field fluctuation in vacuum E_{vac}



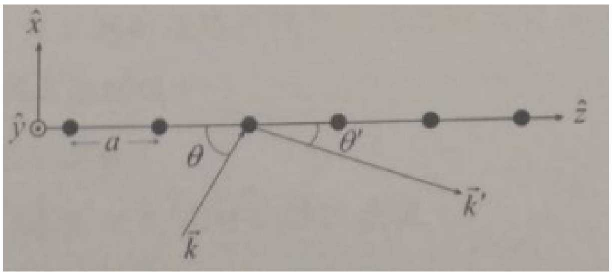
8. In the phasor diagram, the quantum state of the photon satisfies $\Delta X_1 \Delta X_2 \leq \gamma$. Find γ

9. The uncertainty of phasors can also be written as $\Delta n \Delta\phi \geq \beta$. Find β
This inequality means that photons have an uncertainty on particle number n and phase ϕ.

3 Estimated time to solve - 1h30min

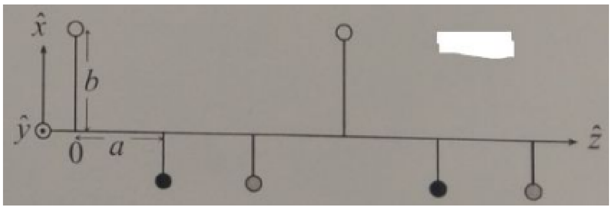
Diffraction pattern of chiral material
Using diffraction pattern of tell left-handed and right handed molecules apart:
We first consider the diffraction pattern of one-dimensional lattice:

1. The figure shows the incident wave (with wave number k), the incident angle is θ , the diffracted wave has wave vector $\vec{k'} = (k, \theta', \Phi')$, find the condition on θ' for constructive interference to happen



If the amplitude of the diffracted wave emitted from each lattice point is proportional to that of the incident wave, and the emitted wave is spherical, then the diffracted wave is $\Psi_{inc}(\vec{R}_m)e^{ik|\vec{r}|-\vec{R}_m}/|\vec{r}-\vec{R}_m| \approx \Psi_{inc}(\vec{R}_m)e^{ik|\vec{r}|-\vec{R}_m}/|\vec{r}|$ when $|\vec{r}| \gg |\vec{R}_m|$ is satisfied. Let $Psi_{inc} = e^{i(\vec{k}\cdot\vec{r}}$, $\vec{R}_m = ma\hat{z}$, find the total diffracted wave Ψ_D from N lattice point

2. Find the condition of constructive interference from Ψ_D
   . Consider the diffraction from one-dimensional left/right-handed symmetric lattice:



The figure on the right shows an one dimensional lattice with right handed symmetry, the small balls represent the atoms, it's the place that can produce diffracted waves. We ignore the diffracting effect of the connecting rods for the atoms, the thin rods have length b , and the neighboring rods are separated by distance a. Also, as the z-coordinate increases, the thin rods rotate relative to each other with angle $\Delta\Phi = \beta\Phi_0$. $\beta = \pm 1$ $\Phi_0 = \pi/3$. The figure shows the case when $\beta = -1$ (corresponding to clockwise rotation), different beta represents different chirality.

3. the position of the atom is $\vec{R}_m = \vec{\rho}_m + ma\hat{z}$, find $\vec{\rho}_m$, β is not yet determined

4. Use $m = 3M + j$, where $j = 0, \pm 1. \vec{k}$ is the same from part 1, $\vec{k'} = (k, \theta', \Phi')$, find Ψ_D. Find the condition of constructive interference from Ψ_D

5. Consider the special case of $\theta = \pi/2$, $a = b = \lambda$, and the condition for constructive interference is fixed at first order, find the relation between the intensity of wave I ($I = |\Psi|^2 = \Psi\Psi^*$) and Φ'.

6. Substitute $\Phi' = \phi + \pi/2$ into $I(\phi)$ and prove that $I(\phi)$ is an even function

7. Discuss the feature of I at $\phi \approx 0$ (or $\Phi' \approx \pi/2$ for $\beta = \pm 1$)

4 Estimated time to solve - 2 hours

Oscillations of the Sun
The sun is made of compressible gas. It can oscillate in a variety of ways. Investigating these oscillations has provided rich information on the interior of the Sun. In this problem we study two kinds of waves: pressure waves and gravity waves.

Part A. Pressure Waves (15 points)
Most of us are familiar with sound waves propagating through Earth's atmosphere, which is a pressure wave. In the Sun, however, we need to consider the fact that gas density falls off with height because of gravity. In this problem, we will use the following notations:

$\bar{m}$ = average mass of particles
g = gravitational acceleration
k_b = Boltzmann constant
T = absolute temperature

γ = ratio of the constant-pressure
specific heat to the constant-volume specific heat

We model the Sun as an atmosphere whose density falls off with height because of gravity. For a thin layer of the atmosphere between heights x and $x+dx$, the equilibrium pressure at these locations are $P(x)$ and $P(x+dx)$ respectively. Assume that the gravitational acceleration and the temperature are constant.

1. Derive the differential equation for the atmospheric density $\rho(x)$

2. The scale height H of the atmosphere is the height through which the density becomes a factor of e^{-1} of the original density. Derive the expression of H.

   When a pressure wave propagates vertically in the atmosphere, the particles will experience small vertical displacements. Let $u(x,t)$ denote the vertical displacement of the gas particles at time t whose undisturbed position is x.

3. As shown in the Fig. 1, there is a change in thickness of the thin layer. Express the change in thickness in terms containing the gradient $\partial u/\partial x$ (Remark: For u being a function of both x and t, $\partial u/\partial x$ is called the partial derivative of u with respect to x with t taken to be constant.)

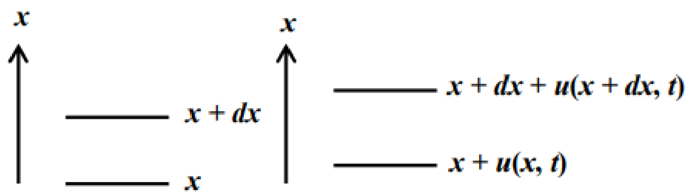


Fig. 1: The vertical displacements of a thin layer of gas particles caused by the propagation of a pressure wave. Note the change in the thickness of the layer.

4. In turn, the vertical displacements produce small fluctuations in density and pressure, denoted as $\delta\rho(x,t)$ and $\delta P(x,t)$ respectively. Express the change in $\delta\rho(x,t)$ and $\delta P(x,t)$n terms containing the gradient $\partial u/\partial x$. Assume that the heat transfer is negligible during the period of the pressure wave.

5. Derive the differential equation of motion for $u(x,t)$. Simplify your expressions using the speed of sound $c_s = \sqrt{\gamma P/\rho}$

6. Show that the solution of the equation of motion is equivalent to a pressure wave traveling through a uniform medium when the wavelength is shorter than a length scale. Derive this length scale.

   Next we seek a sinusoidal wave with angular frequency ω. The energy density of the wave,$\frac{1}{2}\rho\omega^2u^2$ is expected to remain constant as the wave propagates upward with constant velocity in the direction of decreasing density $\rho(x)$. With this expectation in mind we let

$$u(x,t) = \frac{f(x)}{\sqrt{\rho(x)}}e^{-i\omega t}$$

7. Derive the differential equation for f(x).

8. When the frequency of the pressure wave is below a critical frequency ω_c below the Sun's surface, it becomes trapped inside the Sun. What is ω_c?

Part B. Gravity Waves (7 points)

In Part A we only included the restoring force due to the fluctuation in the pressure gradient for pressure waves traveling in the vertical direction of the Sun's atmosphere. However, for gravity waves propagating in a horizontal direction of the Sun's atmosphere, the buoyancy of the gas may also give rise to a restoring force which can sustain oscillations.

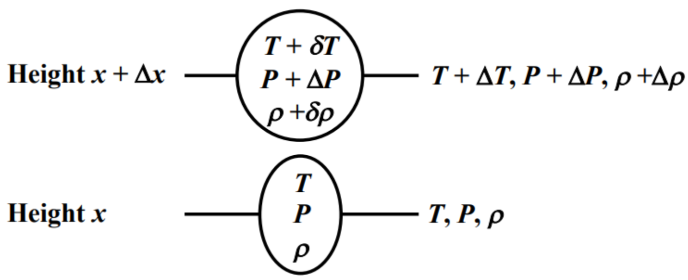


Fig. 2: Displacement of a pocket of gas from height x to height $x + \Delta x$.

To understand this, we consider a small vertical displacement of a pocket of gas in an environment of the same gas with both gradients in the temperature and pressure. As shown in Fig. 2, this pocket of gas has the same temperature, pressure and density as the surrounding gas. When its height is displaced by Δx, it enters an environment with temperature, pressure and density given by $T+\Delta T$, $P+\Delta P$ and $\rho+\Delta\rho$ respectively.

For the pocket of gas, the pressure inside the pocket responds rapidly to the environment so that its pressure also changes by ΔP. On the other hand, the change in temperature and density may be different. Suppose the temperature, pressure and density of the pocket of gas in the new environment are $T + \delta T$, $P + \delta P$ and $\rho + \delta\rho$ respectively. Assume that there is insufficient time for heat conduction from the pocket of gas to the environment.

1. Express $\Delta\rho$ and $\delta\rho$ in terms of expressions containing ΔT and ΔP

2. Suppose the temperature and pressure gradients of the surrounding gas are dT/dx and dP/dx respectively. Derive the equation of motion of the pocket of gas.

3. Determine the range of temperature gradient dT/dx in which the pocket of gas can exhibit oscillations. Express the bound(s) of the temperature gradient in terms of T/H.

4. How does the gas in the Sun behave when the temperature gradient is outside range considered in B3?

5

Problem 5.9

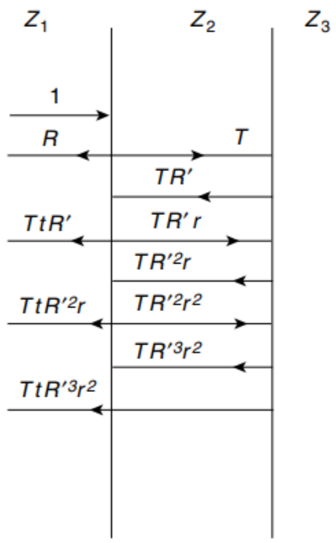
In the figure, media of impedances Z_1 and Z_3 are separated by a medium of intermediate impedance Z_2 and thickness $\lambda/4$ measured in this medium. A normally incident wave in the first medium has unit amplitude and the reflection and transmission coefficients for multiple reflections are shown. Show that the total reflected amplitude in medium 1 which is

$$R + tTR'(1 + rR' + r^2R'^2 \ldots)$$

is zero at $R = R'$ and show that this defines the condition

$$Z_2^2 = Z_1 Z_3$$

(Note that for zero total reflection in medium 1, the first reflection R is cancelled by the sum of all subsequent reflections.)



6

Problem 5.15

The phase velocity v of transverse waves in a crystal of atomic separation a is given by

$$v = c\left(\frac{\sin(ka/2)}{(ka/2)}\right)$$

where k is the wave number and c is constant. Show that the value of the group velocity is

$$c \cos \frac{ka}{2}$$

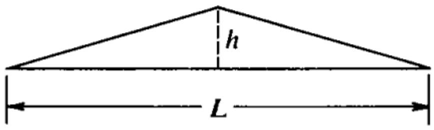
What is the limiting value of the group velocity for long wavelengths?

7

6–12 **A string of length L, which is clamped at both ends and has a tension T, is pulled aside a distance h at its center and released.**

(a) What is the energy of the subsequent oscillations?

(b) How often will the shape shown in the figure reappear? (Assume that the tension remains unchanged by the small increase of length caused by the transverse displacements.)



8

Problem 6.8

In Problem 1.10 we showed that a mass M suspended by a spring of stiffness s and mass m oscillated simple harmonically at a frequency given by

$$\omega^2 = \frac{s}{M + m/3}$$

We may consider the same problem in terms of standing waves along the vertical spring with displacement

$$\eta = (A \cos kx + B \sin kx) \sin \omega t$$

where $k = \omega/v$ is the wave number. The boundary conditions are that $\eta = 0$ at $x = 0$ (the top of the spring) and

$$M\frac{\partial^2\eta}{\partial t^2} = -sL\frac{\partial\eta}{\partial x} \quad \text{at } x = L$$

(the bottom of the spring). Show that these lead to the expression

$$kL \tan kL = \frac{m}{M}$$

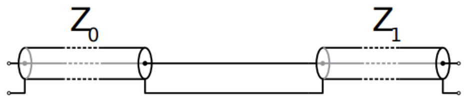
and expand tan kL in powers of kL to show that, in the second order approximation

$$\omega^2 = \frac{s}{M + m/3}$$

The value of v is given in Problem 6.7.

9

An electromagnetic wave can propagate in a transmission line in two opposite directions. For each direction of propagation, the characteristic impedance Z_0 can be used to relate the voltage V_0 and current I_0 amplitudes as in the Ohm's law, $Z_0 = V_0/I_0$. Consider an interface between two transmission lines, with characteristic impedances Z_0 and Z_1. A schematic diagram of the circuit is shown below



Circuit diagram of a transmission line of impedance Z_0 connected to a transmission line of impedance Z_1. The physical size of the interface is much smaller than the wavelength.

When a signal V_i sent into the transmission line with impedance Z_0 reaches the interface it is partially transmitted into the second transmission line, resulting in a signal V_t in that line which propagates forward. Some of the signal may also be reflected, resulting in a backward propagating signal in the initial transmission line V_r.

1. Find the reflectance of the interface $\eta = V_r/V_i$
2. State the condition(s) for the signal V_i to have gained a π phase change on reflection.

10

Problem 4 (35 points)**.** Refer to Figure 4.1. A taut string of length L is placed along the x-axis, whose left end is located at the origin. Both ends of the string can be attached to a vibration generator, which drives oscillations in the y-direction. The speed of wave propagation is u.

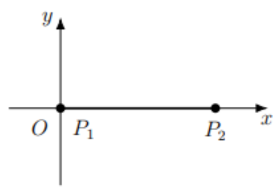


Figure 4.1: A vibrating string.

(1) (22 points). We fix the right end of the string P_2 and connect the left end P_1 to the generator. When the system reaches a steady state, the displacement of the left end is given by $y(x=0,t) = A_0\cos(\omega t)$, where A_0 and ω are the amplitude and angular frequency of the oscillation respectively.

(i) (10 points). We are given that the transverse oscillation attenuates down the string with coefficient $\gamma > 0$. Find the oscillation amplitude everywhere on the string, given that, for a string of infinite length, the equation of a transverse wave travelling and attenuating in the positive x direction is given by $y(x,t) = Ae^{-\gamma x}\cos(\omega t - \omega x/u + \varphi)$, where A and φ are the amplitude and initial phase of the oscillation at $x = 0$ respectively.

(ii) (12 points). We now ignore the effects of attenuation. Find the equation of the standing wave on the string. Find also the positions of the nodes and antinodes of the standing wave.

(2) (13 points). We connect both ends of the string to the generator, such that the displacements of P_1 and P_2 are given by $y(x=0,t) = A_0\cos\omega t$ and $y(x=L,t) = A_0\cos(\omega t + \varphi_0)$ respectively. Ignoring the effects of attenuation, find the equation of the wave everywhere on the string for the cases $\varphi_0 = 0$ and $\varphi_0 = \pi$ respectively, and state the condition for the resonance frequency ω in each case.

11

8–7 (a) As was developed in the text [Eq. (7–12) p. 212], the velocity of sound in a gas is proportional to the square root of the absolute temperature T. Use this fact, and the result of the previous problem, to show that when a thermal gradient exists in the vertical direction (z) sound waves will be turned initially with a radius of curvature

$$R = \frac{2T}{dT/dz}$$

(b) On a still day, the temperature of the atmosphere is found to decrease more or less linearly with height. Sketch the paths of "rays" of sound emitted from a source suspended high in the atmosphere. Assuming that the velocity of sound at ground level is 1100 ft/sec, estimate the horizontal distance at which an airplane flying at 15,000 ft first becomes audible to an observer on the ground, if the temperature decreases by 1° C per 500-ft increase in altitude.

12

8–13 A source of sound of frequency ν_0 moves horizontally at constant speed u in the x direction at a distance h above the ground. An observer is situated on the ground at the point $x = 0$; the source passes over this point at $t = 0$.

(a) Show that the signal received at any time t_R at the ground was emitted by the source at an earlier time t_S, such that

$$\left(1 - \frac{u^2}{v^2}\right) t_S = t_R - \frac{1}{v}\left[h^2\left(1 - \frac{u^2}{v^2}\right) + u^2 t_R^{\,2}\right]^{1/2}$$

(b) Show that the frequency of the received signal, as a function of the *emission* time t_S, is given by

$$\nu(t_S) = \frac{\nu_0}{1 + \dfrac{u}{v} \cdot \dfrac{u t_S}{(h^2 + u^2 t_S^2)^{1/2}}}$$

(The expression for ν as a function of the *reception* time t_R is considerably more complicated.)